Effective analysis of damage spreading in Ising models

1993 J. Phys. A: Math. Gen. 26 L599

(http://iopscience.iop.org/0305-4470/26/14/002)

J. Phys. A: Math. Gen. 26 (1993) L599–L604. Printed in the UK

LETTER TO THE EDITOR

Effective analysis of damage spreading in Ising models

Dietrich Stauffer

Department of Physics, St F X University, Antigonish, Nova Scotia, B2G 1CO, Canada

Received 12 May 1993

Abstract. Using large lattices and different methods of simulation and analysis, the kinetic critical exponent z for damage spreading is shown to be consistent with $z = 2$ in two to five dimensions, without the logarithmic problems seen in some earlier work.

The method of damage spreading, introduced for genetics by Kauffman [1], has for some time been used for Ising magnets [2]. With proper definitions, it was shown to give the static [3] and the dynamic [4] spin correlation functions. Nevertheless, only by assuming logarithmic correction factors could the fractal dimension of the damage cloud be reconciled with the usual fractal dimension $d_f = d - \beta/\nu$ of critical phenomena in d dimensions. And for the dynamics [6], some of the simulations gave a kinetic critical exponent z somewhat higher than most of the recent quality studies by other methods [7,8].

To study damage spreading, we simulate two lattices using the same initial conditions and the same random numbers in a heat-bath procedure, flipping one spin at a time. Only in some localized regions are the two lattices initially different. This perturbation then may, or may not, spread through the lattice as the damage caused by this localized difference. The damage cloud is then the not necessarily connected set of sites differing in a spin-by-spin comparison of the two lattices at a given moment in time, and the damage is the number of sites in this damage cloud, i.e. the Hamming distance. We fix here one (hyper-) plane of one of the lattices as spin up and let all other hyperplanes fluctuate according to the Monte Carlo procedure.

Recent simulations of time-dependent Ising properties [7,8] different from damage spreading showed that one does not have to wait for equilibrium before starting to measure characteristic times. Instead, one may start with all spins parallel, and can observe during the first hundred Monte Carlo sweeps through the lattice that the magnetization decays with time t as $t^{-\beta/z\nu}$, with the usual notation and the equilibrium values of critical exponents (the characteristic time varies as the zth power of the characteristic length). This trick is also employed in the present work. The program is based on the multi-spin coding for d dimensions on workstations, as published in [8].

The fraction $D(x, t)$ of sites damaged in a plane at a distance x from the permanently damaged plane is expected to scale at the Curie point as

$$D(r, t) = x^{-\beta/\nu} f(x/t^{1/z}) \tag{1}$$

according to [4]. Figure 1 shows that our data are compatible with this standard scaling assumption, using $\beta/\nu = 0.515$, $z = 2.05$, $J/kT_c = 0.221\,656$ in lattices of size 400^3. Plots for different z suggest a probable error of the order of 0.1 in z; similar results were

0305-4470/93/140599+06$07.50

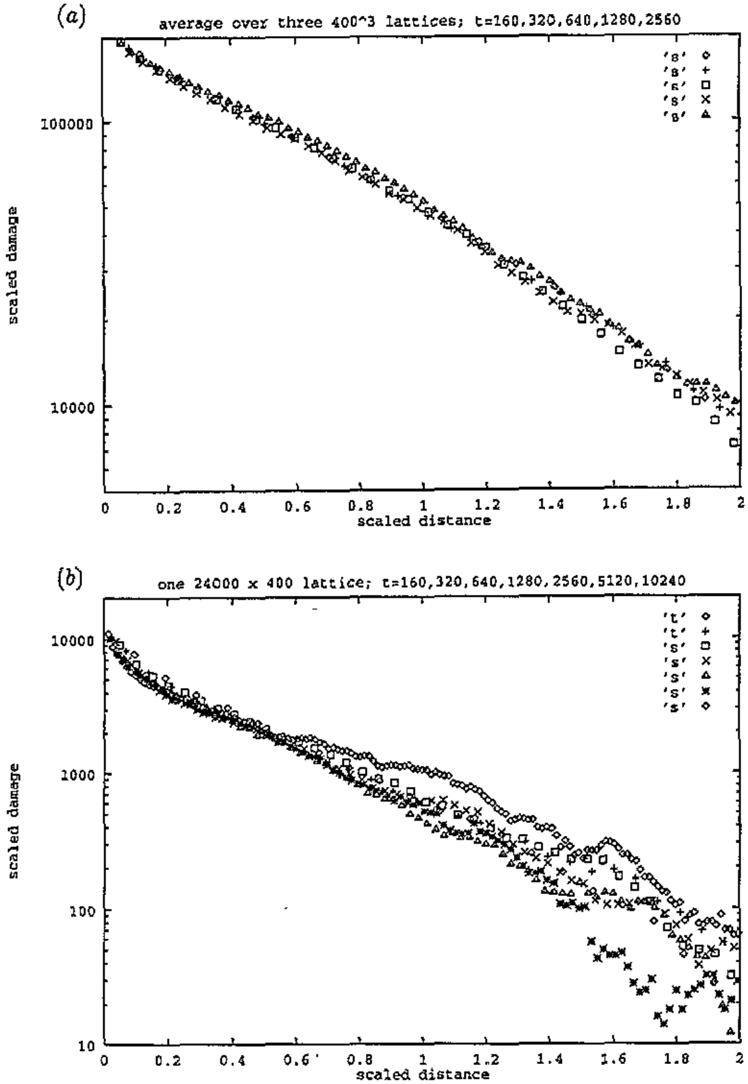

Figure 1. (*a*) Semilogarithmic plot of scaled damage $x^{\beta/\nu}D$ versus scaled distance $x/t^{1/z}$ for $z = 2.05$ in three dimensions. The curves for different times up to 2560 sweeps, represented by different symbols, roughly overlap. (*b*) Same for two dimensions, $z = 2.165$.

found in [4] for two dimensions. Recent work without damage spreading (see [7] for further literature) gave z between 2.03 and 2.10 and thus our z is fully consistent with these data.

The scaled damage $D(x,t)x^{\beta/\nu}$ as a function of distance sinks below some threshold value θ at a distance x proportional to $t^{1/z}$ according to the above scaling law. Figure 2 shows the relation between this threshold distance and time for thresholds $\theta = 0.02$, 0.04, and 0.08. Again, the slopes of these log–log plots give an exponent $1/z$ close to 0.5, without much curvature. Figures 1 and 2 also give two-dimensional results, using $z = 2.165$ [7]. Figure 3 shows analogous plots in four and five dimensions, where our data are roughly consistent with the exact result $z = 2$. (In $4-\epsilon$ dimensions, z remains at 2 to first order in ϵ, and thus $x \propto \sqrt{t}$ *without* logarithmic corrections in four dimensions [9].)

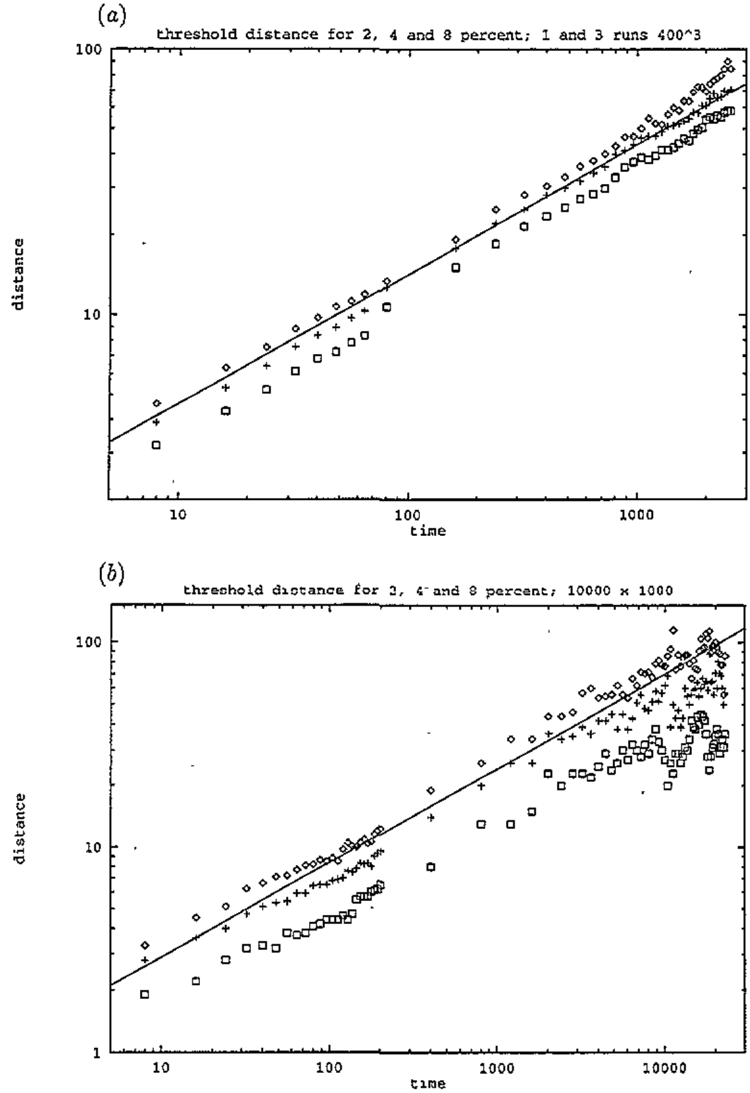

Figure 2. Log–log plot of threshold distances versus time, as evaluated from the data leading to figure 1. For increasing distances, the scaled damage decays below 8 (squares), 4 (crosses), and 2 (diamonds) per cent of a lattice plane at these threshold distances. (*a*) refers to three and (*b*) to two dimensions, the straight lines to $z = 2.05$ and 2.165, respectively.

Thus everything seems fine. Why then were logarithmic and other difficulties observed in [5, 6]? One problem is the use a different criterion for the damage to have reached a certain plane. Whereas here we demand a certain finite threshold value θ for the damage in one plane, in [5, 6] the criterion was to check when the first *single* site was damaged at a given distance or at the lattice boundaries. This criterion may not only give larger fluctuations but also logarithmic correction factors, since for a hyperplane of a hypercubic lattice with L^d spins the criterion for the damage to touch is now

$$L^{d-1} D(x, t) = O(1) .$$

Our semilogarithmic figure 1 shows that at least roughly the damage profile $D(x, t)$ decays

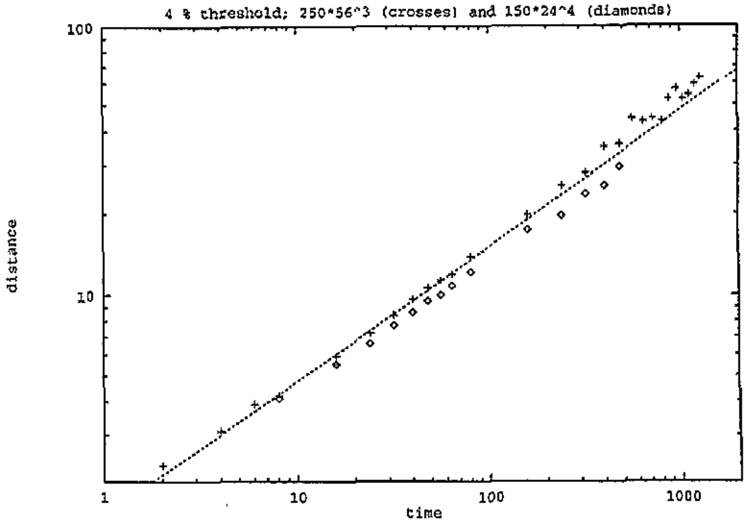

Figure 3. Log–log plot of threshold versus distance for four (diamonds) and five (square) dimensions. The straight line corresponds to $z = 2$. For longer times, constant distances were observed in somewhat smaller lattices.

exponentially with distance x, and then equation (1) gives

$$x \propto t^{1/z} \log(\text{constant} \times L^{d-1} x^{-\beta/\nu}) . \qquad (2)$$

(Similar $\log(N)$ corrections are expected if one would measure the radioactive decay time by observing after what time all N initially radioactive atoms have decayed. Matz *et al* observed the time which an initial damage needs to decay completely [6], and indeed logarithmic correction factors explained their data.)

If we keep the lattice centre permanently damaged and wait until the damage hits the lattice boundaries for the first time, then $x = L/2$, as simulated in [5]. Thus

$$t^{1/z} \propto L/\log(L) \qquad (3)$$

in this method. If in an infinite lattice the total number M of damaged sites increases with time as $t^{d_f/z}$, where $d_f = d - \beta/\nu$ is the usual fractal dimension, then at the time t given by (3) the total damage is

$$M \propto (L/\log(L))^{d_f} \qquad (4)$$

without any multiscaling complications assumed in [5] to explain the numerically observed logarithmic factors. The effective fractal dimension $\mathrm{d}\log(M)/\mathrm{d}\log(t)$ is $d_f(1 - 1/\ln(L))$ according to (4), roughly as found by de Arcangelis *et al* [5]. References [5, 6] were not the only papers employing the logarithmically dangerous method of relying on single sites for the damage touching [10].

Our method of measuring damage spreading in the non-equilibrium part of the relaxation towards equilibrium is valid only for intermediate times and large lattices, $1 \ll t \ll L^z$. If the damage reaches a distance of about $L/2$ at $t \sim L^z$, then with our periodic boundary conditions the tips of the damage clouds start to interfere with each other. Finally, for $t \gg L^z$

the damage profile becomes independent of time and resembles that of the magnetization, as confirmed by simulations. This saturation of the damage means that the above threshold distances no longer increase as $t^{1/z}$ but weaker, leading to an effective exponent z increasing towards infinity due to these finite size effects.

Following [6], we also may first let the system equilibrate for $t \gg L^z$, and only then let the damage spread analogous to [7] for $t \ll L^z$. Now, for the same amount of computer time, the lattices have to be smaller. Indeed, for lattices 40 times smaller than those of figures 1 and 2 for the first few hundred sweeps through the lattice the effective exponent z is close to 2 also in this 'equilibrium' method, but then the damage seems to cross over towards saturation and lets the effective exponent z increase. Similar effects were observed by Hunter *et al* [6]. For larger lattices and shorter times, according to figure 4 these difficulties are avoided and again $z \simeq 2.05$.

(A difference compared with Hunter *et al* [6] is that they let the damage start from a single site, instead of our method of damaging a whole plane. In their method the times for the damage to reach a distance x fluctuate very strongly (see their figure 9). We found the histogram for these times to decay exponentially for long times; thus all moments of this distribution of touching times should give the same characteristic time apart from constant factors. These strong fluctuations thus are not responsible for the strong variation of the effective z observed there. Also, we confirmed their average touching time at $x = 25$ in three dimensions when we let the damage start at the centre site. Thus the difference in the programs, multi-spin coding versus simple storage of spins, should not be blamed for the difficulties.)

In summary, we let the damage spread from the beginning of the simulation and damaged a whole (hyper-)plane. Therefore our lattices were larger and the fluctuations smaller. We found smooth log–log plots of distance versus time leading to reasonable values of z in three-, four- and five-dimensional Ising models. Some logarithmic corrections reported in the literature could be explained and avoided by a better analysis. Thus damage spreading is an intuitively appealing method to study correlations in Ising models. Whether it is more

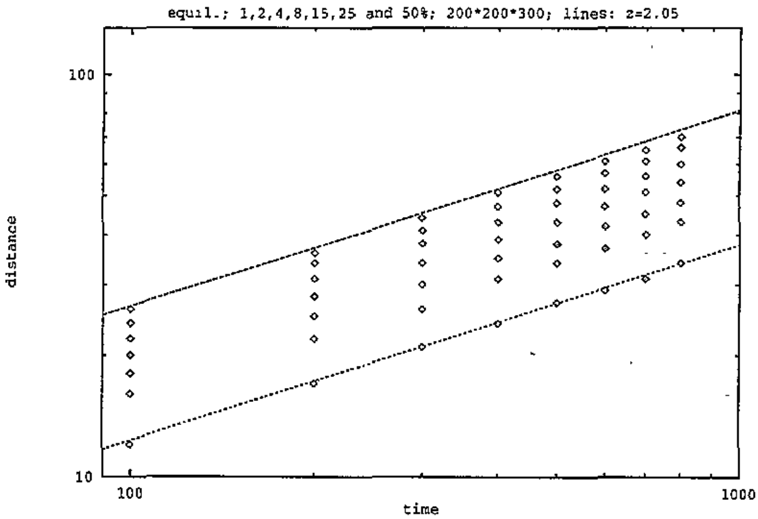

Figure 4. Log–log plot of threshold distances versus time, $\theta = 1, 2, 4, 8, 15, 25$ and 50 per cent (from top to bottom) in three dimensions, when damage starts spreading only after 32 000 sweeps to equilibrate the lattice, followed by 115 samples of 800 sweeps each.

accurate than other methods for determining z remains to be seen; our study employed much less Monte Carlo steps than the work of Ito [7] and Heuer [10].

We thank the Canada Council for support, Naeem Jan and Doug Hunter for many discussions and warm hospitality, and Antonio Coniglio and Hans Herrmann for correspondence.

References

[1] Kauffman S A 1969 *J. Theor. Biol.* **22** 437
[2] Creutz M 1986 *Ann. Phys., NY* **167** 62
[3] Coniglio A, de Arcangelis L, Herrmann H J and Jan N 1989 *Europhys. Lett.* **8** 315
[4] Glotzer S, Poole P H, and Jan N 1992 *J. Stat. Phys.* **68** 895
[5] de Arcangelis L, Coniglio A, and Herrmann H J 1990 *J. Phys. A: Math. Gen.* **23** L265
[6] Poole P H and Jan N 1990 *J. Phys. A: Math. Gen.* **23** L453
Manna S S 1990 *J. Physique* **51** 1261
MacIsaac K and Jan N 1992 *J. Phys. A: Math. Gen.* **25** 2139
Hunter D L, de Arcangelis L, Matz R, Poole P H and Jan N 1993 *Physica* A in press
Matz R, Hunter D L and Jan N 1993 *Preprint*
[7] Ito N 1993 *Physica* **192A** 604; *Physica* A in press, giving earlier literature
Kikuchi M and Ito N 1993 *Preprint*
Adler J 1993 private communication
[8] Stauffer D 1993 *Computer Simulations in Condensed Matter Physics VI* ed D P Landau, K K Mon and H B Schüttler (Berlin: Springer)
[9] Bausch R, Dohm V, Janssen H K, and Zia R K P 1981 *Phys. Rev. Lett.* **47** 1837
[10] Stauffer D 1987 *Phil. Mag.* B **56** 901
[11] Heuer H O 1993 *J. Phys. A: Math. Gen.* **26** L341